Midterm Exam Calculus 2

10 March 2016, 14:00-16:00

university of groningen

The exam consists of 4 problems. You have 120 minutes to answer the questions. You can achieve 100 points which includes a bonus of 10 points.

1. [**8+7+5 Points.**]
   For the function $f : \mathbb{R}^2 \to \mathbb{R}$ with
   $$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases},$$
   (a) use the definition of partial derivatives to calculate $f_x(0,0)$ and $f_y(0,0)$,
   (b) use the definition of the directional derivative to compute $D_{\boldsymbol{v}} f(0,0)$ for a unit vector $\boldsymbol{v} = (u,w) \in \mathbb{R}^2$.
   (c) Is f differentiable at $(x,y) = (0,0)$? Justify your answer.

2. [**10+5+10 Points.**]
   Consider the curve parametrized by $\mathbf{r} : [-1,1] \to \mathbb{R}^3$ with
   $$\mathbf{r}(t) = t\,\mathbf{i} + \frac{1}{3}(1+t)^{3/2}\,\mathbf{j} + \frac{1}{3}(1-t)^{3/2}\,\mathbf{k}.$$
   (a) Determine the parametrization by arc length.
   (b) For each point on the curve, determine a unit tangent vector.
   (c) At each point on the curve, determine the curvature of the curve.

3. [**10+10+5 Points.**]
   (a) Use the method of Lagrange multipliers to find the points (x_1,y_1,z_1) and (x_2,y_2,z_2) on the unit sphere $x^2+y^2+z^2 = 1$ where $f(x,y,z) = x+y-z$ assumes its maximum value and its minimum value, respectively.
   (b) Show that the tangent plane of the unit sphere at the point (x_1,y_1,z_1) is given by the equation $f(x,y,z) = f(x_1,y_1,z_1)$ and the tangent plane of the unit sphere at the point (x_2,y_2,z_2) is given by the equation $f(x,y,z) = f(x_2,y_2,z_2)$.
   (c) Let $(x_0,y_0,z_0) \in \mathbb{R}^3$. Show that f agrees with its linearization at (x_0,y_0,z_0).

4. [**20 Points.**]
   Determine
   $$\iiint_W (x^2+y^2+2z^2)\,\mathrm{d}V,$$
   where W is the solid cylinder defined by the inequalities $x^2+y^2 \leq 4$ and $-1 \leq z \leq 2$.

Solutions

1. (a) Following the definition, the partial derivative of f with respect to x at $(x, y) = (0, 0)$ is

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h\cdot 0}{\sqrt{h^2+0^2}} - 0}{h} = \lim_{h\to 0} 0 = 0.$$

Similarly

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0\cdot h}{\sqrt{0^2+h^2}} - 0}{h} = \lim_{h\to 0} 0 = 0.$$

(b) Let $\boldsymbol{v} = (u, w) \in \mathbb{R}^2$ be a unit vector. Then

$$\begin{aligned} D_{\boldsymbol{v}} f(0,0) &= \lim_{t\to 0} \frac{f((0,0)+t\boldsymbol{v}) - f(0,0)}{t} = \lim_{t\to 0} \frac{f((0+tu, 0+tw)) - f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{\frac{tu\cdot tw}{\sqrt{t^2u^2+t^2w^2}} - 0}{t} = \lim_{t\to 0} \frac{t^2uw}{t^2\sqrt{u^2+w^2}} = \frac{uw}{\sqrt{u^2+w^2}} = uw, \end{aligned}$$

where in the last equality we used that $\boldsymbol{v}$ has unit length.

(c) f is not differentiable at $(0, 0)$. If f was differentiable at $(0, 0)$ then the directional derivative in (b) would be

$$D_{\boldsymbol{v}} f(0,0) = \nabla f(0.0) \cdot \boldsymbol{v} = f_x(0,0)u + f_y(0,0)w = 0$$

for all u, w with $u^2 + w^2 = 1$ as according to (a) we have $f_x(0, 0) = f_y(0, 0) = 0$. This contradicts the result in (b) for $u, w \neq 0$.

2. (a) The tangent vector

$$\mathbf{r}'(t) = \mathbf{i} + \frac{1}{2}(1+t)^{1/2}\,\mathbf{j} - \frac{1}{2}(1-t)^{1/2}\,\mathbf{k}$$

has length

$$\|\mathbf{r}'(t)\|| = \big(1 + \frac{1}{4}(1+t) + \frac{1}{4}(1-t)\big)^{1/2} = \sqrt{\frac{3}{2}}\,.$$

The arc length is hence

$$s(t) = \int_{-1}^{t} |\mathbf{r}'(\tau)|\,\mathrm{d}\tau = \sqrt{\frac{3}{2}}(t+1).$$

Note that $s(-1) = 0$ and $s(1) = \sqrt{6}$ where the latter is the length of the curve. Inverting for t gives

$$t(s) = \sqrt{\frac{2}{3}}s - 1.$$

The parametrization by arc length is hence given by

$$\tilde{\mathbf{r}}(s) = \mathbf{r}(t(s)) = \big(\sqrt{\frac{2}{3}}s - 1\big)\,\mathbf{i} + \frac{1}{3}\big(\sqrt{\frac{2}{3}}s\big)^{3/2}\,\mathbf{j} + \frac{1}{3}\big(2 - \sqrt{\frac{2}{3}}s\big)^{3/2}\,\mathbf{k}\,.$$

(b) The unit tangent vector is given by

$$\mathbf{T} \;=\; \frac{\mathrm{d}\tilde{\mathbf{r}}(s)}{\mathrm{d}s} = \sqrt{\frac{2}{3}}\,\mathbf{i} + \frac{1}{2}\sqrt{\frac{2}{3}}\Big(\sqrt{\frac{2}{3}}s\Big)^{1/2}\mathbf{j} - \frac{1}{2}\sqrt{\frac{2}{3}}\Big(2-\sqrt{\frac{2}{3}}s\Big)^{1/2}\mathbf{k}$$

which agrees with

$$\mathbf{T} = \frac{1}{\|\mathbf{r}'(t)\|}\mathbf{r}'(t)$$

for $t = \sqrt{\frac{2}{3}}s - 1$.

(c) The curvature is given by

$$\begin{aligned}\kappa \;&=\; \left\|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right\| = \left\|\frac{1}{6}\Big(\sqrt{\frac{2}{3}}s\Big)^{-1/2}\mathbf{j} + \frac{1}{6}\Big(2-\sqrt{\frac{2}{3}}s\Big)^{-1/2}\mathbf{k}\right\| \\ &=\; \frac{1}{6}\left(\frac{1}{\sqrt{\frac{2}{3}}s} + \frac{1}{2-\sqrt{\frac{2}{3}}s}\right)^{1/2}\end{aligned}$$

which agrees with

$$\left\|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t}\right\|\frac{1}{\|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\|}$$

for $t = \sqrt{\frac{2}{3}}s - 1$.

3. (a) Let $g(x,y,z) = x^2+y^2+z^2$. Then the unit sphere is the level set of g with value 1. At an extremum of f under the constraint $g(x,y,z) = 1$ there is according to the theorem on Lagrange multipliers a $\lambda \in \mathbb{R}$ such that $\lambda\nabla f(x,y,z) = \nabla g(x,y,z)$. Together with the constraint $g(x,y,z) = 1$ this gives the following four scalar equations:

$$\begin{aligned}\lambda f_x(x,y,z) &= g_x(x,y,z),\\ \lambda f_y(x,y,z) &= g_y(x,y,z),\\ \lambda f_z(x,y,z) &= g_z(x,y,z),\\ x^2+y^2+z^2 &= 1\end{aligned}$$

i.e.

$$\begin{aligned}\lambda &= 2x,\\ \lambda &= 2y,\\ -\lambda &= 2z,\\ x^2+y^2+z^2 &= 1.\end{aligned}$$

We see that $x = y = -z$ which needs to be satisfied together with $x^2+y^2+z^2 = 1$ (λ is then given by, e.g., $2x$). This leads to the two points

$$(x_1,y_1,z_1) = \Big(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\Big)$$

and

$$(x_2,y_2,z_2) = \Big(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\Big)\,.$$

From the Weierstrass Extreme Value Theorem we know that f assumes its maximum and minimum values on the unit sphere. From $f(x_1,y_1,z_1) = \sqrt{3}$ and

$f(x_2, y_2, z_2) = -\sqrt{3}$ we see that (x_1, y_1, z_1) is the point where f assumes its maximum and (x_2, y_2, z_2) is the point where f assumes its minimum.

(b) The tangent plane of the unit sphere at (x_k, y_k, z_k) is orthogonal to $\nabla g(x_k, y_k, z_k)$ for $k = 1, 2$. The tangent plane at (x_k, y_k, z_k) is hence given by $\nabla g(x_k, y_k, z_k) \cdot (x - x_k, y - y_k, z - z_k) = 0$. For (x_1, y_1, z_1) this gives

$$
\begin{aligned}
& 2\big(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\big) \cdot \big(x - \frac{1}{\sqrt{3}}, y - \frac{1}{\sqrt{3}}, z + \frac{1}{\sqrt{3}}\big) = 0 \\
\Leftrightarrow \quad & \frac{1}{\sqrt{3}}x - \frac{1}{3} + \frac{1}{\sqrt{3}}y - \frac{1}{3} - \frac{1}{\sqrt{3}}z - \frac{1}{3} = 0 \\
\Leftrightarrow \quad & x + y - z = \sqrt{3}.
\end{aligned}
$$

As $\sqrt{3} = f(x_1, y_1, z_1)$ we see that the tangent plane of the unit sphere at (x_1, y_1, z_1) satisfies $f(x, y, z) = f(x_1, y_1, z_1)$.

Similarly for (x_x, y_x, z_x) then tangent plane is given by

$$
\begin{aligned}
& 2\big(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\big) \cdot \big(x + \frac{1}{\sqrt{3}}, y + \frac{1}{\sqrt{3}}, z - \frac{1}{\sqrt{3}}\big) = 0 \\
\Leftrightarrow \quad & -\frac{1}{\sqrt{3}}x - \frac{1}{3} - \frac{1}{\sqrt{3}}y - \frac{1}{3} + \frac{1}{\sqrt{3}}z - \frac{1}{3} = 0 \\
\Leftrightarrow \quad & x + y - z = -\sqrt{3}.
\end{aligned}
$$

As $-\sqrt{3} = f(x_2, y_2, z_2)$ we see that the tangent plane of the unit sphere at (x_2, y_2, z_2) satisfies $f(x, y, z) = f(x_2, y_2, z_2)$.

(c) The linearization of f at (x_0, y_0, z_0) is given by

$$
\begin{aligned}
L(x, y, z) &= f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) \\
&= x_0 + y_0 - z_0 + 1 \cdot (x - x_0) + 1 \cdot (y - y_0) - 1 \cdot (z - z_0) \\
&= x + y - z
\end{aligned}
$$

which agrees with $f(x, y, z)$.

4. The cylinder geometry suggests to use cylinder coordinates, i.e. $x = r\cos\theta$, $y = r\sin\theta$ and z stays z. Then

$$
\begin{aligned}
\iiint_W (x^2 + y^2 + 2z^2)\,\mathrm{d}V &= \int_{-1}^{2}\int_{0}^{2}\int_{0}^{2\pi} \big((r\cos\theta)^2 + (r\sin\theta)^2 + 2z^2\big)\, r\mathrm{d}\theta\mathrm{d}r\mathrm{d}z \\
&= 2\pi \int_{-1}^{2}\int_{0}^{2} \big(r^3 + 2z^2 r\big)\,\mathrm{d}r\mathrm{d}z \\
&= 2\pi \int_{-1}^{2} \big[\frac{1}{4}r^4 + z^2 r^2\big]_{r=0}^{r=2}\,\mathrm{d}z \\
&= 2\pi \int_{-1}^{2} (4 + 4z^2)\,\mathrm{d}z \\
&= 2\pi \big[4z + \frac{4}{3}z^3\big]_{z=-1}^{z=2} \\
&= 2\pi\big((8 + \frac{32}{3}) - (-4 - \frac{4}{3})\big) \\
&= 2\pi\big(12 + \frac{36}{3}\big) \\
&= 48\pi.
\end{aligned}
$$